Assignment 7. Due March 10

1. Here we examine an important component of the applicability of the Boussinesq approximation. Consider a dry, adiabatic, hydrostatic atmospheric layer at rest in which the scale of vertical motions is d. The mean quantities in this layer are $T_0(z)$,$p_0(z)$, $\rho_0(z)$ and the derived quantities $\theta_0(z)$ and $\pi_0(z) \equiv [p_0(z)/\hat{p}]^\kappa$ where the notation $\hat{\xi} \equiv \xi(z=0)$ and $\kappa = R_d/c_p$.

   (a) Write the fluid equations for the mean quantities $T_0(z)$,$p_0(z)$, $\rho_0(z)$

   (b) Find and sketch the mean quantities (as functions of z) in terms of their values at $z=0$ and of $H = c_p\theta_0/g$, the "scale height" of the layer

   (c) For what range of values of d/H is this mean atmosphere likely to be a useful approximation to reality.

2. The following problem illustrates some features of entrainment and detrainment in convective clouds. (Ref. Bretherton and Smolarkiewicz. J. Atmos. Sci. 46, 740-759.

   Consider that the cloud is embedded in a hydrostatic, stable environment in which $\frac{d\theta_0}{dz} = \Gamma > 0$, and $\frac{g\Gamma}{\theta_0} = constant = N^2$, where N is the buoyancy frequency of the atmosphere (about 0.01 Hz).

   The "latent heating" distribution in the cumulus cloud is

$$\dot{\mathcal{H}}(x,z,t) = \frac{D\theta}{Dt} = A\sin\left(\frac{z}{z_0}\right)\delta(x)\,H(t) \tag{1}$$

   where, in general, $H(x)$ is the Heaviside step function: $H(x) = 0,\ x < 0$ and $H(x) = 1, \geq 0$. $\frac{dH}{dx} = \delta(x)$, the Dirac delta function.

   $\dot{\mathcal{H}}$ induces two-dimensional motion $u(x,z,t)$,$w(x,z,t)$, pressure perturbation $p*(x,z,t)$ and buoyancy $b(x,z,t)$. The mean atmosphere is at rest.

   (a) Sketch the latent heating profile in the cloud, and describe its implications for buoyancy.

   (b) Write the linearized Boussinesq equations relating u^*, w^*, p^*, θ^* and $\dot{\mathcal{H}}$. Neglect the Coriolis and viscosity terms and set $\frac{dw}{dt} = 0$ in the equation of motion.

   (c) Reduce the equations (hint: take derivatives) to a single differential equation for $b(x,z,t)$, by eliminating first p^* then u^*, then w^* from your equations. Show that

$$b(x,z,t) = b_0(x,t)\sin\left(\frac{z}{z_0}\right)$$

   where

$$\frac{\partial^2 b_0}{\partial t^2} - c^2\frac{\partial b_0}{\partial x^2} = B\delta(x)\,\delta(t) \tag{2}$$

   and identify B and c

(d) The solution to Eq. 2 can be written

$$b_0(x,t) = \frac{B}{2c}\left[H(x+ct) - H(x-ct)\right] \quad (3)$$

From Eq. 3, derive $u^*(x,z,t^0)$ and $w^*(x,z,t^0)$ for some $t^0 > 0$. Sketch these. How are inflow, outflow and buoyancy on the axis related? Describe qualitatively what is going on in this model of the cloud. Bonus points for guessing at some numbers - choose your variety of convective cloud. In particular, what is the lifetime of air in the cloud? Is this shorter, faster, or about the same as what might be a common expectation?